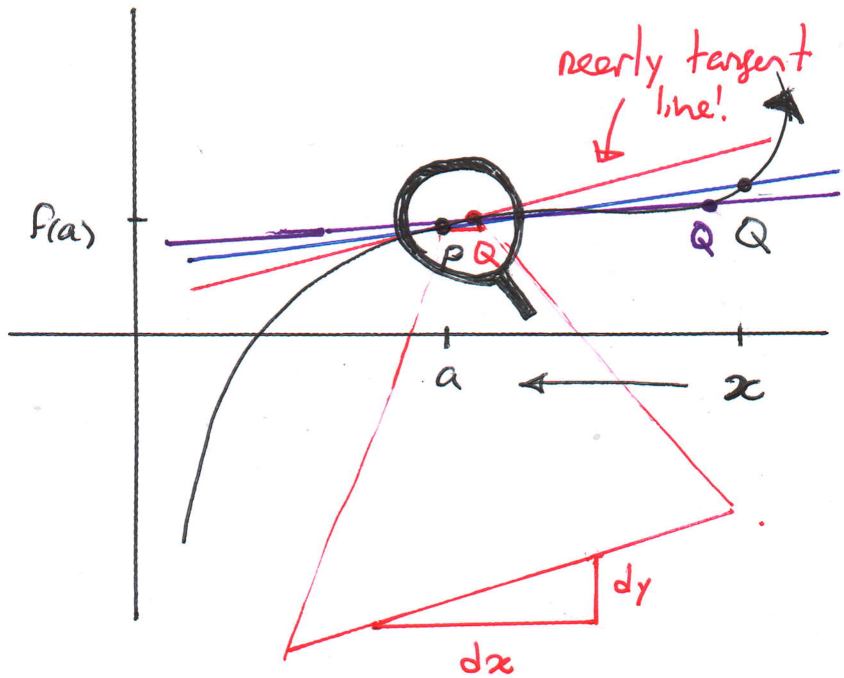
Derivatives Lecture 2

The derivative as a function

If $f(x)$ is a function, then $f'(a) = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$ represents the slope of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$.



$$f'(a) = \frac{dy}{dx}$$

The limit $f'(a)$ is called the derivative at a

There are several useful notations for the derivative

1) $f'(a) = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$

2) $f'(a) = \lim\limits_{x \to a} \dfrac{f(\overbrace{\boxed{x-a}}^{h} + a) - f(a)}{\underbrace{\boxed{x-a}}_{h}} = \lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$

3) (Leibniz Notation) $f'(a) = \left.\frac{df}{dx}\right|_{x=a} = \left.\frac{dy}{dx}\right|_{x=a}$

All of these notations are mathematically but not psychologically equivalent. For instance, notation #3 reminds us of the derivative's humble origin as rise over run. (Just because the derivative moved to the Hamptons, it is not any less a mere peasent! The common slope). It will do you well to remember its proper place. Indeed, the quarrel between Newton and Leibniz over who was the first to discover calculus has lead to the boycotting of Leibniz notation in Britain. This slowed the progress of calculus on the island for 50 years!

Ex. Let $f(x) = -2x^2 + 5x + 13$. Compute $f'(a)$ using

(a) Notation #1

(b) Notation #2

which notation do you think is better for this problem?

Solution:

(a) $$f'(a) = \lim_{x \to a} \frac{(-2x^2+5x+13)-(-2a^2+5a+13)}{(x-a)}$$

$$= \lim_{x \to a} \frac{-2(x^2-a^2)+5(x-a)}{(x-a)}$$

$$= \lim_{x \to a} \frac{-2(x+a)(x-a)+5(x-a)}{(x-a)}$$

$$= \lim_{x \to a} \frac{[-2(x+a)+5]\cancel{(x-a)}}{\cancel{(x-a)}} = -2(a+a)+5$$

$$= -4a+5.$$

(b) $$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{\left(-2(a+h)^2 + 5(a+h) + 13\right) - \left(-2a^2 + 5a + 13\right)}{h}$$

$$= \lim_{h \to 0} \frac{-2(a^2 + 2ah + h^2) + 5a + 5h + 2a^2 - 5a}{h}$$

$$= \lim_{h \to 0} \frac{-\cancel{2a^2} + \cancel{2a^2} - 4ah + 5h + \cancel{5a} - \cancel{5a} - 2h^2}{h}$$

$$= \lim_{h \to 0} \frac{-4ah + 5h - 2h^2}{h} = \lim_{h \to 0} \left(-4a + 5 - 2h\right)$$

$$= -4a + 5$$

Observe that Notation #2 psychologically compelled us to expand the parentheses and distribute terms. The formulas were bulkier and they used up more symbols. Notation #2 was leading us to the result via a more tedious and lengthier route.

<u>Remark:</u> Notation #1 is much better for all algebraic functions. For example if $f(x) = \dfrac{x^2+\sqrt{5x^4+1}}{x^2+4}$, then

$$(1)\quad f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$$

$$= \lim_{x \to a} \frac{\dfrac{x^2+\sqrt{5x^4+1}}{x^2+4} - \dfrac{a^2+\sqrt{5a^4+1}}{a^2+4}}{x-a}$$

This expression alone uses 33 symbols.

$$(2)\quad f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$$

$$= \lim_{h \to 0} \frac{\dfrac{(a+h)^2+\sqrt{5(a+h)^4+1}}{(a+h)^2+4} - \dfrac{a^2+\sqrt{5a^4+1}}{a^2+4}}{h}$$

This expression uses 43 symbols

with each step using Notation #2, you will write about 10 exta symbols (and actually much more!) A little bit of thinking can save a lot of time!

Recall that if $f(x) = x^2$, then $f'(a) = 2a$ and if $f(x) = \sqrt{x}$, $f'(a) = \frac{1}{2\sqrt{a}}$.

$f(x)$ is a function of x, whereas $f'(a)$ is a function of a. It is sometimes convenient to express f and f' as functions of the same variable:

<u>Def:</u> Let $y = f(x)$ be a function. Then the corresponding derivative function is $y = f'(x)$.

where

$$(1)\quad f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$

or

$$(2)\quad f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Ex. Let $f(x) = x^2 + 3\sqrt{9-x}$. Find $f'(x)$.

Solution:

$$f'(x) = \lim_{z \to x} \frac{\left(z^2 + 3\sqrt{9-z}\right) - \left(x^2 + 3\sqrt{9-x}\right)}{(z-x)}$$

$$= \lim_{z \to x} \frac{z^2 - x^2}{z-x} + 3\lim_{z \to x} \frac{\sqrt{9-z} - \sqrt{9-x}}{z-x}$$

$$= \lim_{z \to x} \frac{(z+x)\cancel{(z-x)}}{\cancel{(z-x)}} + 3\lim_{z \to x} \frac{\left(\sqrt{9-z} - \sqrt{9-x}\right)\left(\sqrt{9-z} + \sqrt{9-x}\right)}{(z-x)\left(\sqrt{9-z} + \sqrt{9-x}\right)}$$

$$= 2x + 3\lim_{z \to x} \frac{(9-z) - (9-x)}{(z-x)\left(\sqrt{9-z} + \sqrt{9-x}\right)}$$

$$= 2x + 3\lim_{z \to x} \frac{-\cancel{(z-x)}}{\cancel{(z-x)}\left(\sqrt{9-z} + \sqrt{9-x}\right)}$$

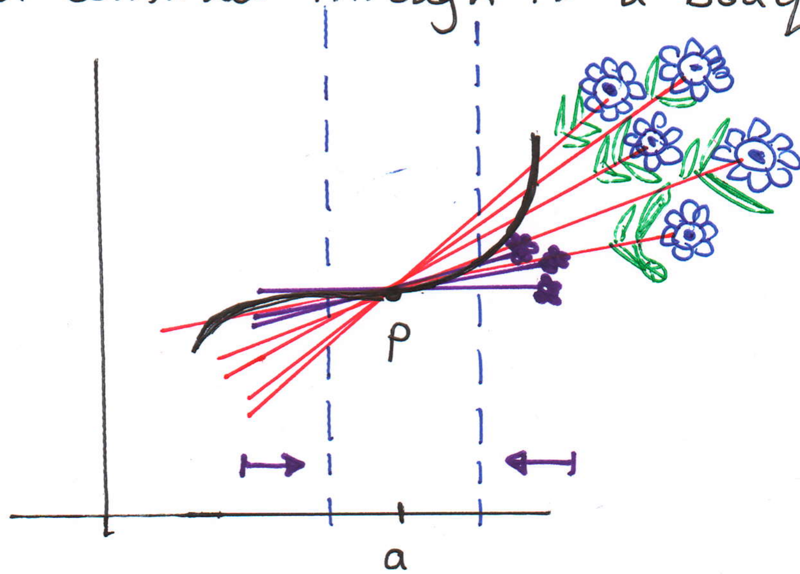
$$= 2x + \frac{-3}{2\sqrt{9-x}}$$

Relationship between $y = f(x)$ and $y = f'(x)$

You have observed this relationship countless times! From space, the earth appears to be shaped like a ball, but zoom in and it seems flat. What accounts for this effect?

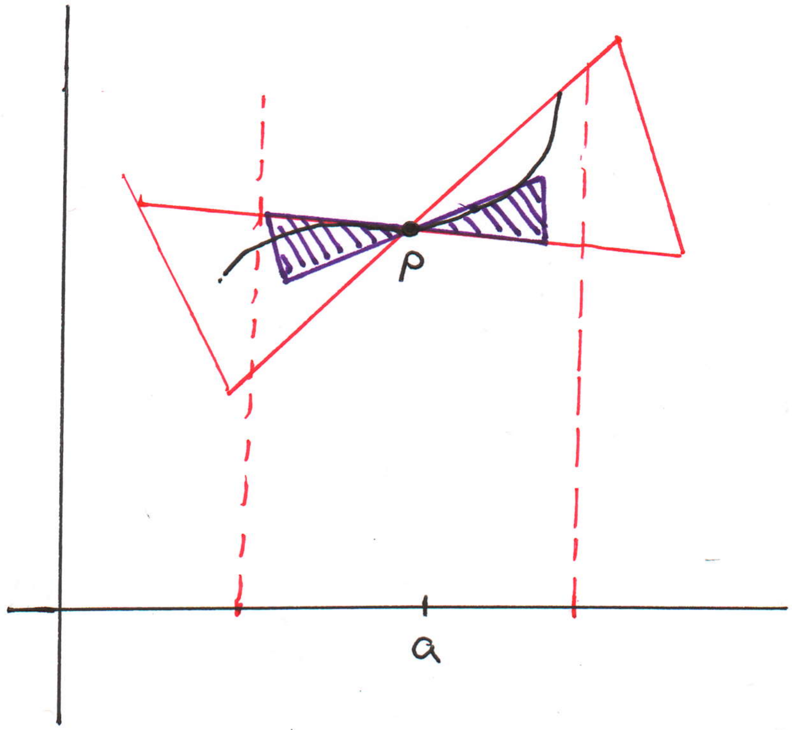
Zoom in on the curve $y = f(x)$ at $(a, f(a))$. What shape will you see? Well, if $f'(a)$ exists the shape you will eventually start to see is a line!

To notice that this is indeed so, imagine some curve and construct through it a bouquet of lines.



The bouquet consists of all lines that cross the point $P = (a, f(a))$ and some other point

On the graph $y = f(x)$. There are so many lines in fact, that the bouquet looks like a solid cone.



As we zoom in at $(a, f(a))$, the values of x reside closer and closer to a and the angle at which the triangles of the cone meet at P tends to 0. In other words, all the possible lines at P start to look like the one line. Namely the tangent line.

In other words, $y = f(x)$ looks like the line

$$y = f(a) + f'(a)(x - a).$$

If you are clever, you will be able to recognize many important features of a differentiable function at a glance.

<u>Observation:</u> $y = f(x)$ differentiable means the graph looks locally like a line!

<u>Thm:</u> If $f(x)$ is differentiable at a, then $f(x)$ is continuous at a.

<u>Proof:</u> First, this result is obvious. If we zoom in at $x = a$ (i.e. at $(a, f(a))$) we will see a line. Now a line is connected. Hence the function is continuous at $x = a$.

The formulas show us just as much! Hint: How would you write $\lim_{x \to a} f(x)$ in terms of $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$?

$$\lim_{x \to a} f(x) = \lim_{x \to a} f(a) + \left(f(x) - f(a)\right)$$

$$= \lim_{x \to a} \left(f(a) + \frac{f(x) - f(a)}{(x-a)} (x-a) \right)$$

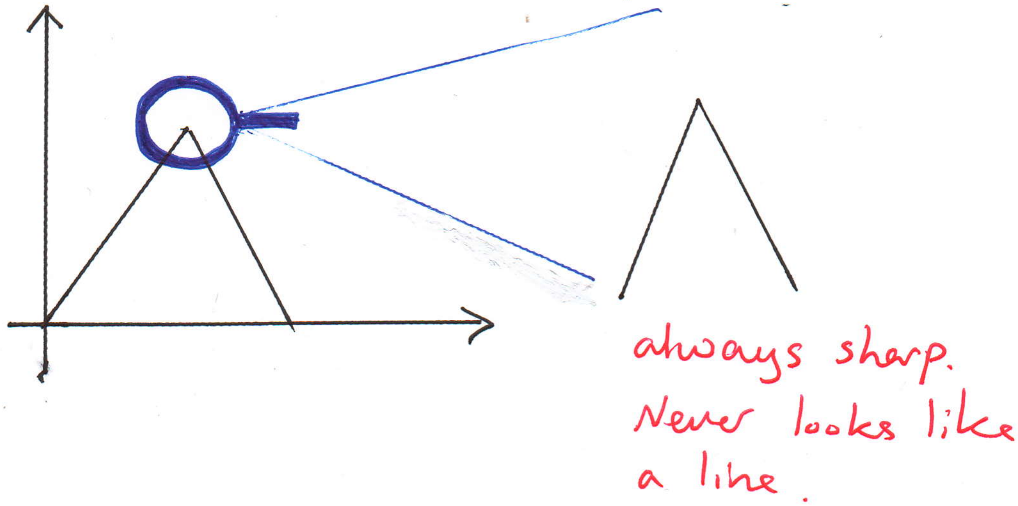
$$= f(a) + \left(\lim_{x \to a} \frac{f(x) - f(a)}{x - a} \right) \left(\lim_{x \to a} (x-a) \right)$$

$$= f(a) + f'(a) \cdot 0 = f(a).$$

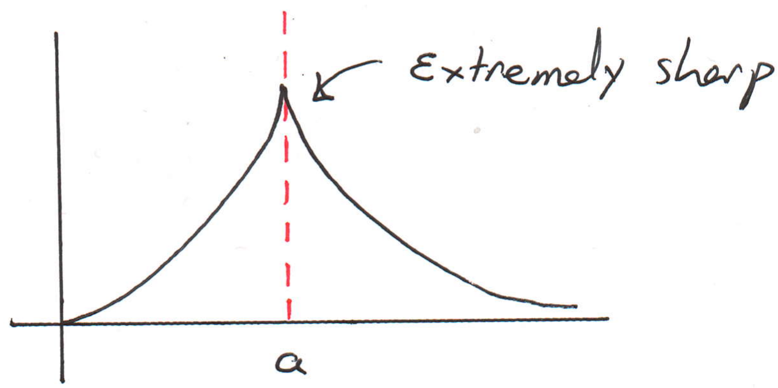
Thus $\lim_{x \to a} f(x) = f(a)$.

Q. Differentiability implies continuity. Is the opposite true? Does continuity imply differentiability?

A. Asked in another way, the question is: If we zoom in on a continuous curve, will it start to look like a straight line? We can clearly imagine that the answer is no.



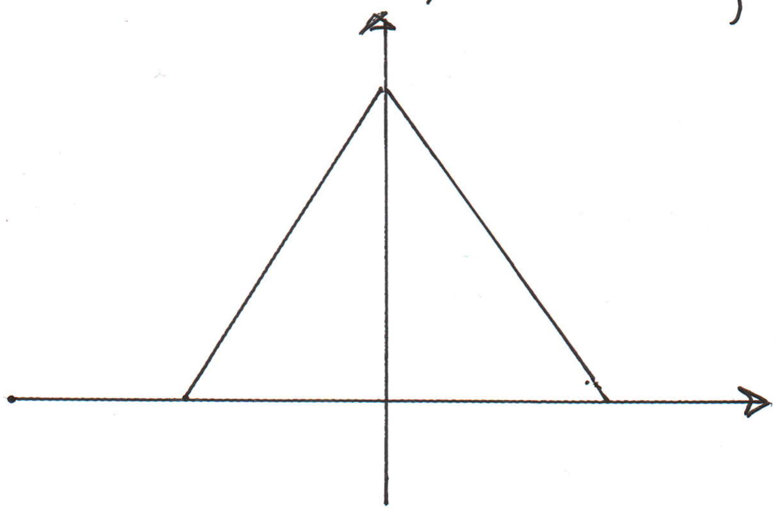
Differentiable curves are also called smooth. They are pleasent to the touch, because you don't cut your your finger when you run it along the curve.



$$\lim_{x \to a^-} \frac{f(x) - f(a)}{x - a} = +\infty, \quad \lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} = -\infty.$$

Ex. Let $f(x) = 1 - |x|$ where does this function fail to be differentiable?

Solution: Observe first that $f(x)$ is everywhere continuous (How are you observing it?)



The function is defined by joining together 2 pieces

$$f(x) = \begin{cases} 1+x & \text{if } x<0 \\ 1-x & \text{if } x \geq 0 \end{cases}$$

Each piece looks like a line segment and is therefore differentiable. Thus the only possible point of non-differentiability is at $x=0$.

Now, $\lim\limits_{x \to 0^+} \dfrac{f(x)-f(0)}{x-0} = \lim\limits_{x \to 0^+} \dfrac{(1-x)-1}{x}$

$$= \lim_{x \to 0^+} \frac{-x}{x} = -1$$

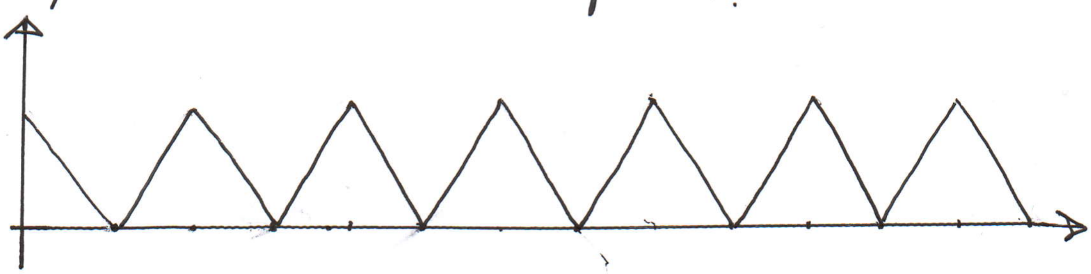
whereas

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{(1+x) - 1}{x}$$

$$= \lim_{x \to 0^-} \frac{x}{x} = 1$$

Since left and right limits arent equal the derivative does not exist.

We can easily produce graphs with infinitely many non-differentiable points.



For instance, if $f(x)$ = shortest distance from x to an integer, the graph $y = f(x)$ will look like this

Notice that if we move a little to the left or a little to the right of a point where this function is not differentiable (i.e. doesn't locally look like a line) we see a point for which $f(x)$ is differentiable.

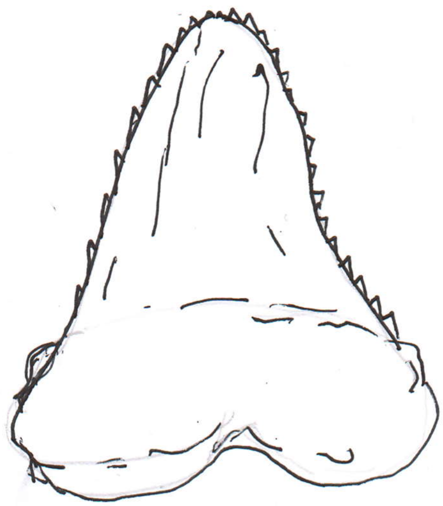
Q. Are all continuous functions essentially smooth? In other words, if you move from a sharp point a little to the left or a bit to the right, will the graph look smooth?

A. To me the answer was obvious, because of my days working at the New York Aquarium by the shark tank. Children used to come and bother me with stupid questions:

Oh a shark! Does it bite?

Oh yes my little one, it bites!

At this point I would show them the Megalodon tooth



It is a tooth covered by many tiny sharp teeth. Zoom in on one mini-tooth and you will see that it is also covered in teeth etc.

As in Bertolt Brecht's song:

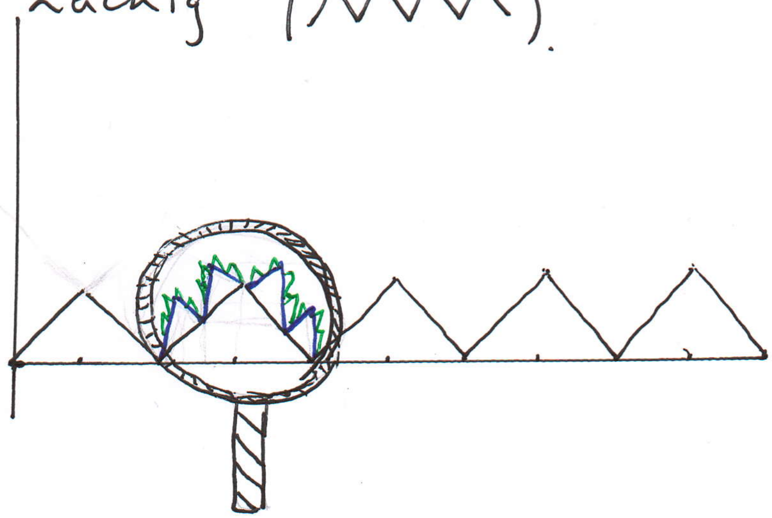
Und der Haifisch, der hat Zähne,
Und die trägt er im Gesicht...

Oh the shark, babe, has such teeth, dear
And it shows them pearly white.

We can construct such a tooth function out of $f(x)$ = distance from x to the closest integer:

$$g(x) = \sum_{n=1}^{\infty} \frac{f(2^n x)}{2^n}$$

This just means: Impose more tiny teeth on every tooth. As the Germans say, make it "zackig" (/\/\/\/\).



Comprehension Check: Without doing any computations Find the derivative of $f(x) = 5x+2$ at $x = -7$.

Solution:

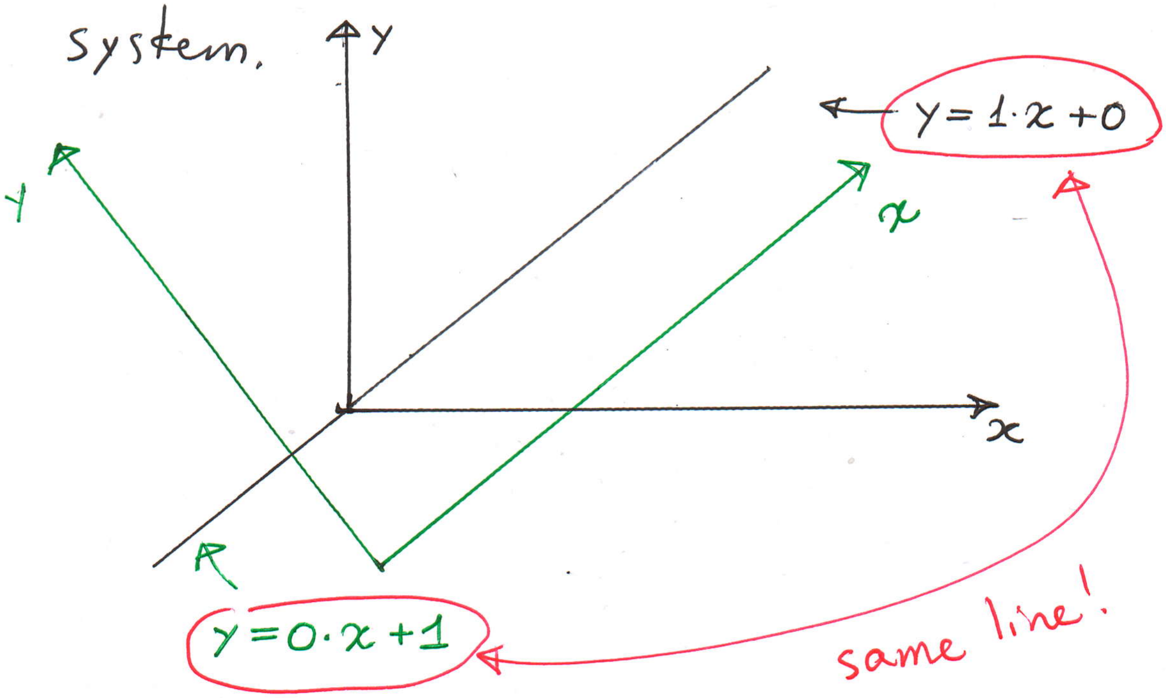
The derivative is the slope of the tangent line. Thus $f'(-7)$ is the slope of the line that the graph $y = f(x)$ resembles under very heigh zoom on the point $(-7, f(-7))$.

But $f(x) = 5x+2$ is a line with slope 5!!!

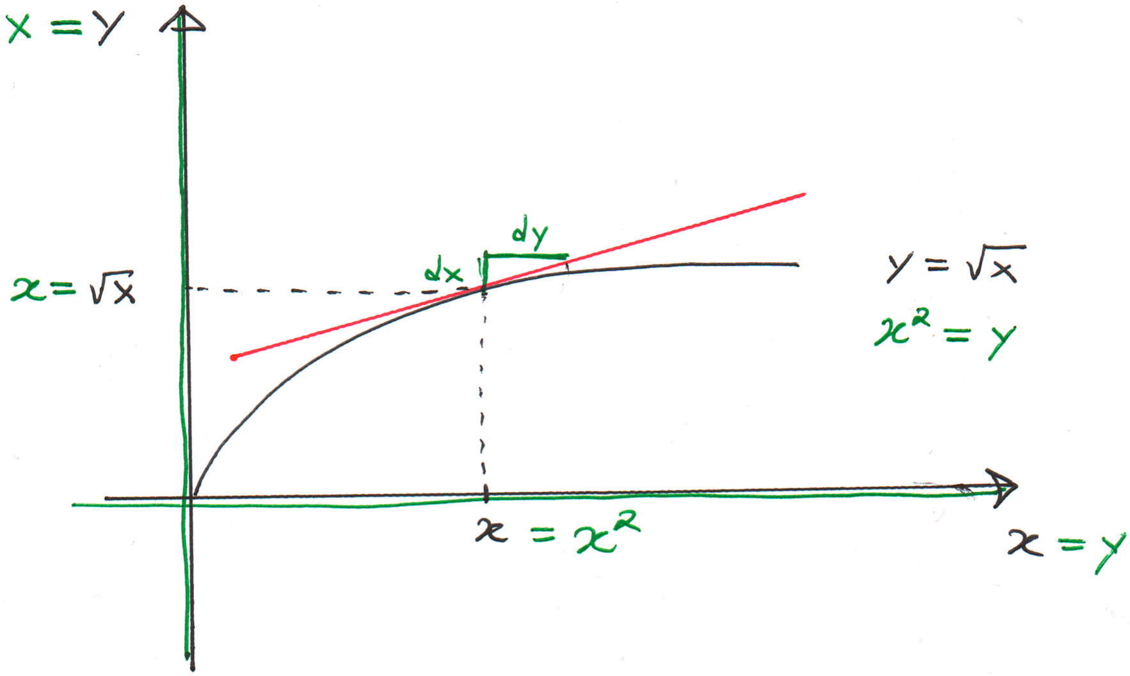
Thus $f'(-7) = 5$. In fact, $f'(x) = 5$ for all x!

In short, if you zoom in on a line it looks like the same line.

__Remark:__ The geometrically significant feature of differentiable (or smooth) curves is that when you zoom in, the curve starts to resemble a line. The slope is not an intrinsic feature of a line, but rather that of the coordinate system.



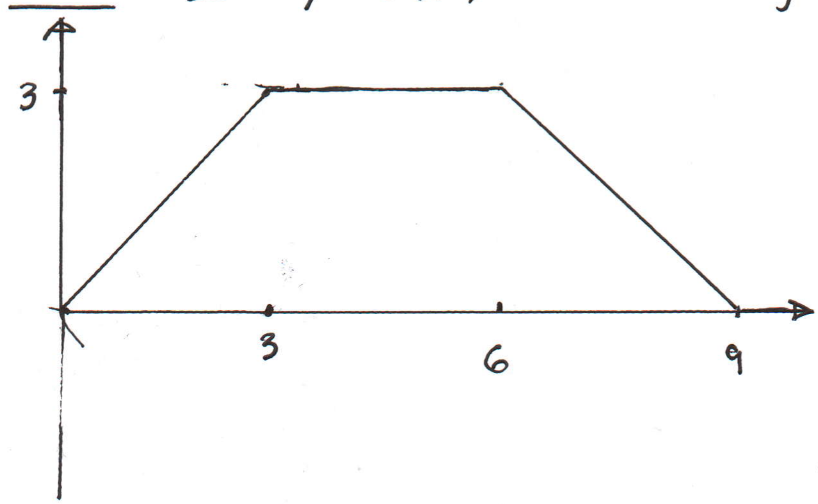
In fact, you will redefine the derivative as the tangent line, once you reach multivariable calculus. This idea is very useful. The curves $y = \sqrt{x}$ and $y = x^2$, for instance, have the same shape. and consequently the derivative formulas for $f(x) = x^2$ and $g(x) = \sqrt{x}$ must be related.



Slope of the tangent line relative to the black axis is $\frac{dy}{dx} = \frac{1}{\frac{dy}{dx}} = \frac{1}{2x} = \frac{1}{2\sqrt{x}}$

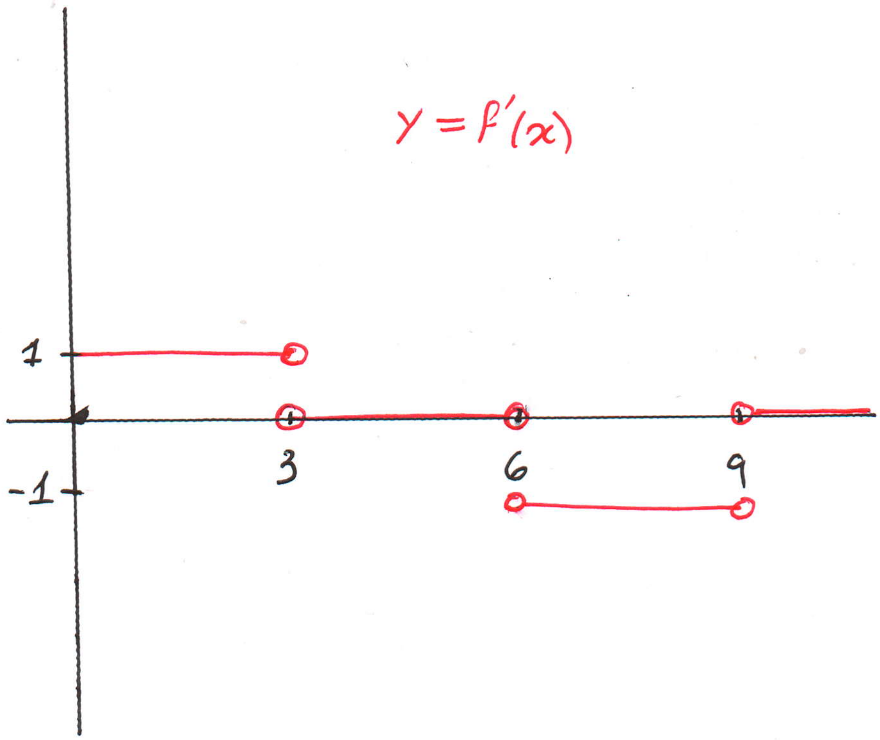
If this is confusing, don't worry! We will explore this idea a bit later.

<u>Ex.</u> Let $y = f(x)$ be the graph



Plot the graph $y = f'(x)$

Solution: Simply zoom in on the graph at some particular point. What line do you see? what is its slope?



To see this graph, simply note that from 0 to 3, we climb linearly $((0,0) \to (3,3))$ so the slope is

$\frac{3-0}{3-0} = 1$. Next the line joining $(3,3)$ to $(6,3)$ is horizontal, so the slope is 0. Further to the right

The line segment joins (6, 3) to (9, 0). Hence its slope is $\frac{0-3}{9-6} = \frac{-3}{3} = -1$.

After that, the curve is a horizontal line on the x-axis.

It is easy in this fashion to plot the derivative of a polygonal curve (e.g. curve built up from line segments).

To draw other curves, simply imagine them to be chains of short line segments.